The motion of pairs of gas bubbles in a perfect liquid

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SUMMARY

In a systematic construction of a theory for bubbly liquids, one encounters the problem of the interaction between two spheres in a perfect liquid. This paper is devoted to that problem for the case in which the motion stems from the instantaneous acceleration of the liquid in which the spheres are immersed. Trajectories described by their separation vector in the course of time are numerically computed with use of the analytically obtained flow potential. An approximate theory is developed from which qualitative properties of these trajectories are obtained. The effect of the relative motion on the pair distribution in e.g., a bubbly flow is considered as well.

1. Introduction

The problem considered in this paper has to do with the theory of flow of inhomogeneous media, in particular with the flow of particles or bubbles dispersed in a liquid of small viscosity. The theoretical approach is, like in the kinetic theory of gases, to carry out successive approximations in terms of the volume density of the inhomogeneities. In the lowest approximation one considers the particle to be isolated in the liquid. The interaction, which is (and here, of course, is the difference with the kinetic theory of gases) of a purely hydrodynamic nature, is neglected in this approximation. In the next approximation the interaction with just one other particle is allowed for. The outlines of such a theory for mixtures of liquid and small gas bubbles is given in Van Wijngaarden [1]. The problem of the hydrodynamics of two particles moving in a perfect liquid appears as an element in the theory. The present paper deals with that problem and is, for convenience, restricted to rigid spheres of zero mass. For liquid in which there are no surface-active agents and in which the relative motion is dominated by inertial effects, this is a reasonable model for the real flow (for a review of properties of such mixtures, see e.g., Van Wijngaarden [2]). The equation of motion for each of the spheres in such a pair is

$$\int_A p d\mathbf{A} = 0, \tag{1.1}$$

in which p is the pressure and $d\mathbf{A}$ an element of the surface of the sphere over which the integration is carried out. The pressure p is connected with the potential ϕ of the flow by Bernoulli's Theorem, which is (t being time and ρ the liquid's density):

$$\frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - \tfrac{1}{2}\{|\nabla\phi|\}^2. \tag{1.2}$$

So the question is how to obtain the flow potential ϕ.

To be specific we shall consider the problem of two spheres, separated by a distance R, immersed in an infinite perfect liquid, which is at rest at times $t < 0$. At $t = 0$ the liquid is accelerated, for example, by a piston, and we ask for the subsequent motion of the two spheres. For equal velocities the potential for two spheres has been obtained by Jeffrey [3] in terms of twin spherical expansions. In that form the potential has been used by Van Wijngaarden [4] to calculate the motion right after the instantaneous acceleration. Then only the term linear in ϕ in (1.2) counts. For $t > 0$ also the quadratic term is important which complicates severely the problem of finding the motion from (1.1) and (1.2). The construction of the potential for the motion of two spheres with unequal velocities is done in Section 2. The derivation of the equations governing the mean motion and the relative motion is presented in Section 3. In Section 4 we consider the trajectory described by the separation vector $\mathbf{R}$ in the course of time after the instantaneous acceleration discussed above. Numerical computation of this is possible with help of the results of Section 3. A number of results is shown in Section 4. In view of the complexity of the equations, it is rather difficult to interpret and explain these results. Qualitative insight is obtained by considering a simplified system of equations in which, from the singularities, only the dipoles are taken into account, higher-order singularities being disregarded. Albeit even then analytical solution does not appear possible, a number of important results regarding the properties of the trajectories are obtained. In Section 5 where this is done these features are also compared with solutions of the exact equations, dealt with in Section 4.

Finally, in Section 6, we consider the problem of finding the probability distribution, as affected by the relative motion.

2. Potential for the motion of a pair of spheres in a perfect liquid

We consider two massless spheres, A and B, of equal radius and immersed in a perfect liquid which is at $t = 0$ instantaneously accelerated to velocity $\mathbf{U}$. As a result of the acceleration of the liquid, the spheres start to move in such a way that there is no resultant force on each of them. Initially only the term $-\partial\phi/\partial t$ (see 1.2) in Bernoulli's Theorem counts, and the spheres acquire a velocity which has the same value for each of them and which is directed along $\mathbf{U}$. This velocity, which has been calculated already in [4], serves as initial condition in the present work in which the motion for $t > 0$ will be investigated.

Since initially the motion is in the direction of $\mathbf{U}$, a plane containing $\mathbf{U}$ and the line connecting the centres will also contain the initial velocities. In view of symmetry, no motion of the spheres will be induced in the direction of the normal to this plane. The liquid velocity, of course, has a component in that direction, however, not the spheres. In the following we will

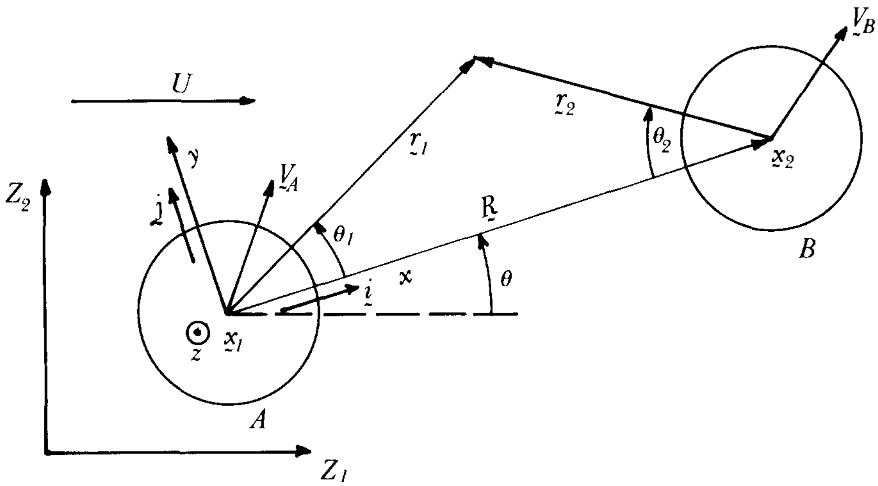

Figure 1. Two spheres A and B moving with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ in reference frames used in the paper.

use several coordinate frames, which are summarized in Figure 1. First there is a Cartesian frame (x, y, z), the x, y-plane of which coincides with the plane through $\mathbf{U}$ and the vector $\mathbf{R}$ directed from the centre sphere A to that of sphere B. The x-axis is along the line of centres. We indicate the unit values in x- and y-direction with $\mathbf{i}$ and $\mathbf{j}$ respectively. Then there are spherical polar coordinate systems (r_1, θ_1, μ) and (r_2, θ_2, μ) connected with the centres $\mathbf{x}$ of A and $\mathbf{x}_2$ of B in the manner shown in Figure 1. Finally, parallel to the x, y-plane, there is a *fixed* plane of reference in which there is a rectangular frame (Z_1, Z_2) and in which the angle subtended between $\mathbf{R}$ and $\mathbf{U}$ is indicated with θ.

The spheres will, after acquiring equal velocities at $t = 0$, obtain different velocities after $t = 0$ because of the interaction stemming from the term $-\frac{1}{2}\{|\nabla\phi|\}^2$ in (1.2). When these velocities are indicated with $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively, the velocity common to both is $\frac{1}{2}(\mathbf{v}_A + \mathbf{v}_B)$ and the relative velocity is $(\mathbf{v}_A - \mathbf{v}_B)$.

Since Laplace's equation, which the potential ϕ has to satisfy, is linear we may superpose solutions for the case in which these velocities are equal and for the case in which they are equal in magnitude but opposite in sign. The former of these has been given by Jeffrey [3] in a study on steady heat conduction involving two spheres and his solution was used by Van Wijngaarden [4] in connection with finding the effective virtual mass of a sphere in a suspension. In both papers the potential is expressed in terms of spherical polar coordinates. We shall start with this form, although we will later need expressions in x, y and z. With

$$\mathbf{G} = \mathbf{U} - \tfrac{1}{2}(\mathbf{v}_A + \mathbf{v}_B) = G_0\mathbf{i} + G_1\mathbf{j} \tag{2.1}$$

an appropriate solution of Laplace's equation is

$$\phi_1 = \mathbf{G}\cdot\mathbf{x} + \sum_{m=0}^{1} \sum_{n=m}^{\infty} G_m \left\{ g_{mn}^{(1)} \left(\frac{a}{r_1}\right)^{n+1} P_n^m(\cos\theta_1) + g_{mn}^{(2)} \left(\frac{a}{r_2}\right)^{n+1} P_n^m(\cos\theta_2) \right\} \cos m\mu \, . \tag{2.2}$$

In this expression the $g_{mn}^{(1)}$ and $g_{mn}^{(2)}$ are coefficients to be determined from the boundary condition on the spheres demanding the vanishing of the relative velocity component in the direction of the normal to the surface. In order to apply this condition on either of the spheres, we have to express spherical harmonics of argument r_1 and θ_1 in harmonics of argument r_2 and ϕ_2. This can be done with the help of the relation (see e.g., Jeffrey [3])

$$\left(\frac{a}{r_i}\right)^{n+1} P_n(\cos\theta_i) = \left(\frac{a}{R}\right)^{n+1} \sum_{q=m}^{\infty} \binom{n+q}{q+m} \left(\frac{r_{3-i}}{R}\right)^q P_q^m(\cos\ \theta_{3-i}). \tag{2.3}$$

Using this relation we find from the boundary conditions

$$-\left(1+\frac{1}{n}\right) g_{mn}^{(i)} + \sum_{q=m}^{\infty} \binom{n+q}{n+m} g_{mq}^{(3-i)} \left(\frac{a}{R}\right)^{n+q+1} = (-1)^{i(m-1)} a\delta_{1n}, \tag{2.4}$$

where δ_{ij} is the Kronecker delta. From symmetry it follows that we may write

$$g_{mn}^{(1)} = (-1)^{m-1} g_{mn}^{(2)} = a g_{mn}. \tag{2.5}$$

Next we write g_{mn} as a series expansion in (a/R),

$$g_{mn} = \frac{1}{2} \sum_{p=0}^{\infty} K_{mnp} \left(\frac{a}{R}\right)^p. \tag{2.6}$$

Inserting this together with (2.5) into (2.4) gives

$$-\left(1+\frac{1}{n}\right) \sum_{p=0}^{\infty} K_{mnp} \left(\frac{a}{R}\right)^p + \sum_{q=m}^{\infty} \binom{n+q}{n+m} \sum_{p=0}^{\infty} (-1)^{m-1} K_{mpq} \left(\frac{a}{R}\right)^{n+q+p+1}$$
$$= 2(-1)^{m-1}\delta_{1n}, \tag{2.7}$$

from which, by collecting like powers of a/R,

$$K_{mn0} = (-1)^m \delta_{1n},$$
$$K_{mnp} = (-1)^{m-1} \frac{n}{n+1} \sum_{q=1}^{1/2[p-n-3]} \binom{n+q}{n+m} K_{mq(p-q-n-1)}. \tag{2.8}$$

Next we consider the case where A and B have velocities of equal magnitude, but opposite in sign. Introducing

$$\mathbf{w} = \tfrac{1}{2}(\mathbf{v}_A - \mathbf{v}_B) = w_0\mathbf{i} + w_1\mathbf{j}, \tag{2.9}$$

we write, in analogy with (2.2),

$$\phi_2 = -\tfrac{1}{2} \sum_{m=0}^{1} \sum_{n=m}^{\infty} w_m \left\{ f_{mn}^{(1)} \left(\frac{a}{r_1}\right)^{n+1} P_n^m(\cos\theta_1) - f_{mn}^{(2)} \left(\frac{a}{r_2}\right)^{n+1} P_n^m(\cos\theta_2) \right\} \cos m\mu. \tag{2.10}$$

Application of the boundary conditions

$$\nabla\phi_2 \cdot \mathbf{n} = \begin{cases} \mathbf{w}\cdot\mathbf{n}, & \text{on sphere } A, \\ -\mathbf{w}\cdot\mathbf{n}, & \text{on sphere } B, \end{cases} \tag{2.11}$$

n denoting a unit vector in the direction normal to the surface, yields with help of (2.3)

$$\left(1 + \frac{1}{n}\right) f_{mn}^{(i)} + \sum_{q=m}^{\infty} \binom{n+q}{n+m} f_{mq}^{(3-i)} \left(\frac{a}{R}\right)^{n+q+1} = -(-1)^{i(m-1)} a \delta_{1n}. \tag{2.12}$$

The antisymmetry expressed by (2.11) allows the simplification

$$f_{mn}^{(1)} = (-1)^{m-1} f_{mn}^{(2)} = a f_{mn},$$

which gives, upon writing

$$f_{mn} = \tfrac{1}{2} \sum_{p=0}^{\infty} L_{mnp} \left(\frac{a}{R}\right)^p \tag{2.13}$$

and substitution in (2.12),

$$L_{mn0} = (-1)^m \delta_{1n},$$

$$L_{mnp} = (-1)^m \frac{n}{n+1} \sum_{q=1}^{1/2[p-n-3]} \binom{n+q}{n+m} L_{mq(p-q-n-1)}. \tag{2.14}$$

For further reference we note that from (2.6) and (2.8) on one hand, and from (2.13) and (2.14) on the other, it follows that

$$f_{mn}\left(\frac{a}{R}\right) = (-1)^{n+1} g_{mn}\left(-\frac{a}{R}\right). \tag{2.15}$$

By combination of (2.2) and (2.10) we find for the potential of the flow around the two spheres A and B, the first moving with velocity $\mathbf{v}_A$ and the second with velocity $\mathbf{v}_B$, the velocity at infinity being **U**,

$$
\begin{aligned}
\phi = \phi_A &= \mathbf{U}\cdot\mathbf{x} + \sum_{m=0}^{1}\sum_{n=m}^{\infty} aD_{mn}\left(\frac{a}{r_1}\right)^{n+1} P_n^m(\cos\theta_1)\cos m\mu \\
&+ \sum_{m=0}^{1}\sum_{n=m}^{\infty} aE_{mn}(-1)^{m-1}\left(\frac{a}{R}\right)^{n+1}\sum_{q=m}^{\infty}\binom{n+q}{n+m}\left(\frac{r_1}{R}\right)^{q} P_q^m(\cos\theta_1)\cos m\mu,
\end{aligned}
\tag{2.16}
$$

when referred to coordinates centred in A. In this expression

$$D_{mn} = G_m g_{mn} - w_m f_{mn}, \tag{2.17}$$

$$E_{mn} = G_m g_{mn} + w_m f_{mn}. \tag{2.18}$$

Likewise, in terms of r_2, θ_2 and μ,

$$
\begin{aligned}
\phi = \phi_B &= \mathbf{U}\cdot\mathbf{x} + \sum_{m=0}^{1}\sum_{n=m}^{\infty} (-1)^{m-1} aE_{mn}\left(\frac{a}{r_2}\right)^{n+1} P_n^m(\cos\theta_2)\cos m\mu \\
&+ \sum_{m=0}^{1}\sum_{n=m}^{\infty} aD_{mn}\left(\frac{a}{R}\right)^{n+1}\sum_{q=m}^{\infty}\binom{n+q}{q+m}\left(\frac{r_2}{R}\right)^{q} P_q^m(\cos\theta_2)\cos m\mu.
\end{aligned}
\tag{2.19}
$$

3. Motion of a pair of identical spheres

The dynamics of a closed body moving through an infinite quiescent liquid can be formulated in terms of a Lagrangian, see e.g., Lamb [5]. The insertion of rigid walls in the flow can be taken into account as well [6]. However, for systems of bodies moving through the liquid, no exact general theory is available because of the complexity of the interactions. For the present problem which involves two bodies, we make use of a result obtained recently by Landweber and Miloh [7] for the force exerted on a body moving through an arbitrary flow in the case in which this flow can be represented by multipoles of given strength.

Let a multipole of order q (a source has order zero, a dipole has order one, etc.) have a strength $\mathbf{M}_q$ and be situated in (x_k, y_k, z_k). Accordingly, when the velocity field $\mathbf{u}$ has a potential ϕ,

$$\mathbf{u} = \nabla\phi, \tag{3.1}$$

the velocity is singular in (x_k, y_k, z_k). We can write therefore $(\mathbf{u})_k$ as the sum of a regular part $(\mathbf{u}_R)_k$ and a singular part $(\mathbf{u}_S)_k$. In addition we introduce the volume τ of the considered body, and we define

$$D_k^q = \frac{\partial^q}{\partial x^\alpha \partial y^\beta \partial z^\gamma}, \qquad \alpha+\beta+\gamma = q, \tag{3.2}$$

$$\sigma = \{(x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2\}^{1/2}. \tag{3.3}$$

The force $\mathbf{F}$ on the body with volume τ and moving with velocity $\mathbf{v}$ in the arbitrary potential flow is, as obtained by Landweber and Miloh [7],

$$\mathbf{F} = \rho \frac{d}{dt} \{\tau \mathbf{v} - 4\pi \sum_k M_q D_k^q (\sigma)_k\} - 4\pi\rho \sum_k M_q D_k^q (\mathbf{u}_R)_k. \tag{3.4}$$

The derivation of (3.4) by Landweber and Miloh [7] contains some errors which, however, do not affect the final result [8]. This is confirmed by a derivation along quite different lines by one of the present authors [9], which gives the same result. To see how singularities in the expressions (2.16) and (2.19) compare with the multipoles $\mathbf{M}_q$, we make use of the cartesian frame (x, y, z) in Figure 1. The relation between these coordinates and the spherical coordinates displayed in Figure 1 is

$$\begin{aligned} x &= r_1 \cos\theta_1 = R - r_2 \cos\theta_2, \\ y &= r_1 \sin\theta_1 \cos\mu = r_2 \sin\theta_2 \cos\mu, \\ z &= r_1 \sin\theta_1 \sin\mu = r_2 \sin\theta_2 \sin\mu. \end{aligned} \tag{3.5}$$

To convert (2.16) and (2.19) in these coordinates, we use the following relations (Morse and Feshbach [10] pp. 1270, 1281)

$$\left(\frac{\partial}{\partial y} \pm i \frac{\partial}{\partial z}\right)^m \frac{\partial^{n-m}}{\partial x^{n-m}} \frac{1}{r_1} = (-1)^{n+m} (n-m)! e^{\pm im\mu} P_n^m(\cos\theta_1),^\dagger \tag{3.6}$$

$$\int_0^{2\pi} X^n \cos mu\,du = (-1)^m \frac{2\pi i^m n!}{(n+m)!} r_1^n P_n^m(\cos\theta_1) \cos m\mu, \tag{3.7}$$

with

$$X = x + iy\cos u + iz\sin u. \tag{3.8}$$

Similar expressions hold for r_2 and θ_2. For those coordinates $\partial/\partial x$ has to be replaced by (see 3.5) $\partial/\partial(R-x) = -\partial/\partial x$. Taking notice of this we can write the expressions (2.16) and (2.19) for ϕ_A and ϕ_B in terms of x, y and z with the help of (3.6)–(3.8):

$$\begin{aligned} \phi_A = \mathbf{U}\cdot\mathbf{x} &+ \sum_{m=0}^{1} \sum_{n=m}^{\infty} \frac{(-1)^{n+m} a^{n+2} D_{mn}}{(n-m)!} \left(\frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial x}\right)^{n-m} \left(\frac{1}{r_1}\right) \\ &- \sum_{m=0}^{1} \sum_{n=m}^{\infty} aE_{mn} \left(\frac{a}{R}\right)^{n+1} \sum_{q=m}^{\infty} \binom{n+q}{q+m} \frac{(q+m)!}{2\pi q! R^q} (i)^{-m} \int_0^{2\pi} X^q \cos mu\,du, \end{aligned} \tag{3.9}$$

† The definition P_n^m used in [10] differs from the usual one (e.g., [12]) which is used throughout this paper.

$$\phi_B = \mathbf{U}\cdot\mathbf{x} - \sum_{m=0}^{1}\sum_{n=m}^{\infty}(-1)^m \frac{a^{n+2}E_{mn}}{(n-m)!}\left(\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial x}\right)^{n-m}\left(\frac{1}{r_2}\right)$$
$$-\sum_{m=0}^{1}\sum_{n=m}^{\infty} aD_{mn}\left(\frac{a}{R}\right)^{n+1}\sum_{q=m}^{\infty}(-1)^q\binom{n+q}{q+m}\frac{(q+m)!}{2\pi q!R^q}(i)^{-m}\int_0^{2\pi}X^q\cos mu\,du. \tag{3.10}$$

In these forms the potentials ϕ_A and ϕ_B consist of a regular part and an infinite sum of singularities. The strength M_q of a singularity, as occurring in (3.4), can be directly read from the above expressions. Since the force on each of the spheres must vanish, the equations of motion are, from (3.4), (3.9) and (3.10),

$$0 = \frac{d}{dt}\left\{\tau\mathbf{v}_A + 3\tau\sum_{m=0}^{1}(-1)^m D_{m1}\begin{pmatrix}\delta_{0m}\\ \delta_{1m}\\ 0\end{pmatrix}\right\} +$$
$$+ 2\pi a^2\sum_{m=0}^{1}\sum_{l=0}^{1}\sum_{n=1}^{\infty}(-1)^{n+m}D_{mn}D_{l(n+1)}n^m(n+2)^{l+1}\begin{pmatrix}2\delta_{0l}\delta_{0m}+\delta_{1l}\delta_{1m}\\ \delta_{1l}\delta_{0m}-\delta_{0l}\delta_{1m}\\ 0\end{pmatrix}, \tag{3.11}$$

$$0 = \frac{d}{dt}\left\{\tau\mathbf{v}_B + 3\tau\sum_{m=0}^{1}(-1)^m E_{m1}\begin{pmatrix}\delta_{0m}\\ \delta_{1m}\\ 0\end{pmatrix}\right\}$$
$$- 2\pi a^2\sum_{m=0}^{1}\sum_{l=0}^{1}\sum_{n=1}^{\infty}(-1)^{n+m}E_{mn}E_{l(n+1)}n^m(n+2)^{l+1}\begin{pmatrix}2\delta_{0l}\delta_{0m}+\delta_{1l}\delta_{1m}\\ \delta_{1l}\delta_{0m}-\delta_{0l}\delta_{1m}\\ 0\end{pmatrix}. \tag{3.12}$$

The coefficients D_{mn} and E_{mn} are related to $\mathbf{G}$ and $\mathbf{w}$ by (2.17) and (2.18). Using these expressions and also (2.1) and (2.9) we find, by addition and subtraction, from (3.11) and (3.12)

$$\frac{d}{dt}\{U_0 + G_0(3g_{01}-1)\} = \frac{3}{2a}\{2G_0w_0(\beta_{00}+\alpha_{00}) + G_1w_1(\beta_{11}+\alpha_{11})\}, \tag{3.13}$$

$$\frac{d}{dt}\{U_1 - G_1(3g_{11}+1)\} = \frac{3}{2a}\{G_1w_0(\beta_{01}-\alpha_{10}) + G_0w_1(\alpha_{01}-\beta_{10})\}, \tag{3.14}$$

$$\frac{d}{dt}\{w_0(1-3f_{01})\} = \frac{3}{2a}\{2G_0^2\zeta_{00} + G_1^2\zeta_{11} + 2w_0^2\eta_{00} + w_1^2\eta_{11}\}, \tag{3.15}$$

$$\frac{d}{dt}\{w_1(1+3f_{11})\} = \frac{3}{2a}\{G_0G_1(\zeta_{01}-\zeta_{10})+w_0w_1(\eta_{01}-\eta_{10})\}. \tag{3.16}$$

In these four equations for G_0, G_1, w_0, w_1 the symbols α, β, η and ζ represent the following:

$$\alpha_{ml} = \sum_{n=1}^{\infty} (-1)^n n^m (n+2)^{l+1} g_{mn} f_{l(n+1)}, \tag{3.17}$$

$$\beta_{ml} = \sum_{n=1}^{\infty} (-1)^n n^m (n+2)^{l+1} f_{mn} g_{l(n+1)}, \tag{3.18}$$

$$\eta_{ml} = \sum_{n=1}^{\infty} (-1)^n n^m (n+2)^{l+1} f_{mn} f_{l(n+1)}, \tag{3.19}$$

$$\zeta_{ml} = \sum_{n=1}^{\infty} (-1)^n n^m (n+2)^{l+1} g_{mn} g_{l(n+1)}. \tag{3.20}$$

The velocity acquired by the spheres at $t = 0^+$, and serving here as initial value, is obtained by putting the left-hand sides of (3.13)–(3.16) equal to zero, giving

$$t = 0: \quad w_0 = w_1 = 0,$$

$$G_0 = \frac{U_0}{1-3g_{01}}, \qquad G_1 = \frac{U_1}{1+3g_{11}}. \tag{3.21}$$

Using (2.1) and (2.9) and, for calculating g_{01} and g_{11}, (2.6) and (2.8), this results in

$$t = 0: \quad v_{A,0} = v_{B,0} = v_0$$

$$= 3U_0 \left\{1+\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{R}\right)^p\right\} \bigg/ \left\{1+3\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{R}\right)^p\right\}, \tag{3.22}$$

$$v_{A,1} = v_{B,1} = v_1 = 3U_1\left\{1-\sum_{p=3}^{\infty} K_{11p}\left(\frac{a}{R}\right)^p\right\} \bigg/ \left\{1-3\sum_{p=3}^{\infty} K_{11p}\left(\frac{a}{R}\right)^p\right\}. \tag{3.23}$$

4. Trajectories of pairs of spheres

The velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ can be obtained from the equations given in the preceding section, with (3.22) and (3.23) as initial values. The velocity $\mathbf{U}$ is constant for $t > 0$. The description of the relative motion can be done in terms of R and θ, as depicted in Figure 1. From this figure and from (2.9), we have

$$\frac{dR}{dt} = -2w_0, \tag{4.1}$$

$$R\frac{d\theta}{dt} = -2w_1. \tag{4.2}$$

For the description of the mean motion the coordinates Z_1 and Z_2 may be used. We want to give here some results of the numerical computation of trajectories from the pertinent equations, given in the previous section.

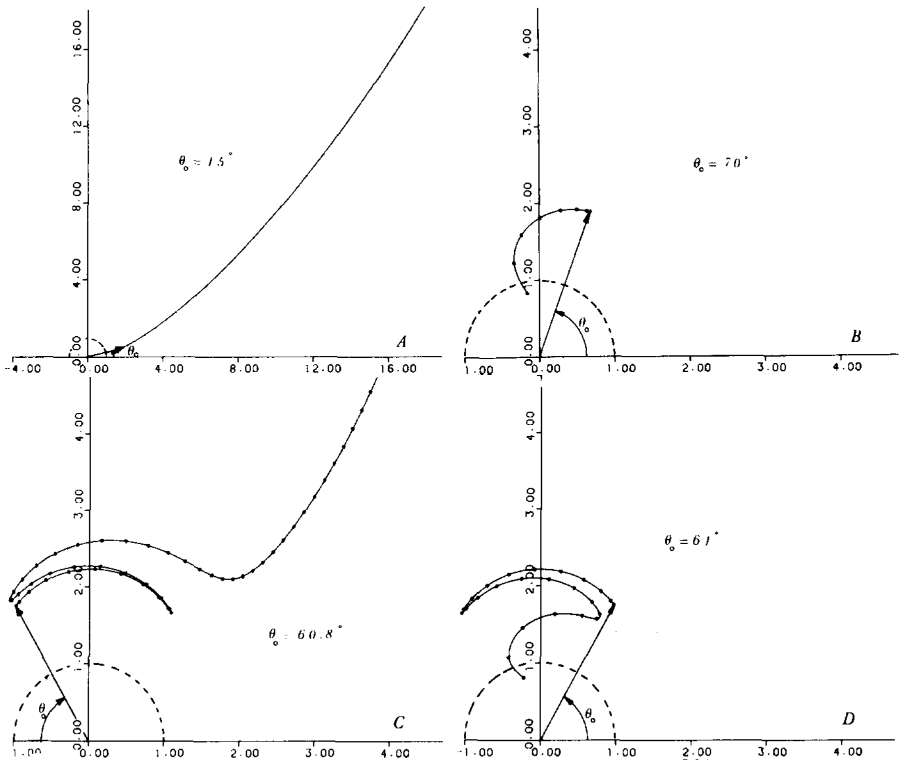

Figure 2. Relative motion of two spheres. The pictures show the separation R and the angle θ for several initial values of θ_0 and $R_0 = 4a$. The unit along the axes represents 2a. (A) two spheres escaping from each other; (B) two spheres attracting each other; (C) oscillating trajectory with eventual escape; and (D) oscillating trajectory with eventual attraction.

In Figure 2A–2D a number of computed trajectories is shown. A point along a trajectory marks the length of R, and the magnitude of the angle θ with the incoming flow direction. The unit along the axes is 2a. Each trajectory starts at $t = 0$. From the numerical calculations follows that three classes of trajectories can be distinguished:

Class I. Characteristic for this class is that R *grows* monotonously from the beginning, while the angle θ, after slowly increasing, reaches eventually a constant value. An example of this class is shown in Figure 2A.

Class II. R decreases monotonously and θ tends towards the value $\pi/2$. An example is given in Figure 2B. Eventually R decreases below 2a which is physically impossible. Hence, a continuation of the trajectory after touching has to be given. For this there are several possibilities. One possibility is to assume that two particles stick together once they touch. Another possibility is to assume that the spheres just pass each other and continue their way.

Class III. The behaviour of the trajectory becomes complicated, according to the numerical computations, when the starting position is under an angle of about 60.5°. The distance R varies only very slowly initially, while the angle θ starts oscillating around $\theta = \pi/2$. Eventually the two spheres in the pair either escape from each other, as for example happens with the pair the trajectory of which is shown in Figure 2C or they come together as in Figure 2D. This complicated behaviour, as illustrated by the above examples, may be understood qualitatively when we ignore in the calculations all the singularities except the dipoles. This approximation is of some additional interest because within the attained accuracy it is possible to determine quantitatively the effect of the relative velocity in a pair of spheres on the probability distribution. This is done in Section 6.

5. Approximate theory for trajectories

With the help of (2.15) and of (2.6)–(2.8), we derive from relations (3.17)–(3.20)

$$\begin{aligned}
\alpha_{00} + \beta_{00} &= O\left(\frac{a}{R}\right)^7, \\
\alpha_{11} + \beta_{11} &= O\left(\frac{a}{R}\right)^7, \\
\alpha_{10} - \beta_{01} &= O\left(\frac{a}{R}\right)^7, \\
\beta_{10} - \alpha_{01} &= O\left(\frac{a}{R}\right)^7, \\
\zeta_{00} = -\zeta_{11} &= \frac{3}{2}\left(\frac{a}{R}\right)^4 + O\left(\frac{a}{R}\right)^7, \\
\zeta_{01} = -\zeta_{10} &= \frac{3}{2}\left(\frac{a}{R}\right)^4 + O\left(\frac{a}{R}\right)^7, \\
g_{01} = f_{01} &= \tfrac{1}{2} + O\left(\frac{a}{R}\right)^3, \\
g_{11} = f_{11} &= -\tfrac{1}{2} + O\left(\frac{a}{R}\right)^3.
\end{aligned} \tag{5.1}$$

From inserting these relations in (3.15) and (3.16), it follows that **w** grows to a magnitude of order $U(a/R)^3$ in a time of order R/U. The right-hand sides of (3.14) and (3.15) then are small of order $(a/R)^9$. Since a/R is at most 1/2, this is an extremely small quantity and it suggests itself to restrict attention to the relative motion assuming the mean motion to be given by its initial value, given in (3.22) and (3.23). Then we have, from (2.1)

$$\mathbf{G} = -2\mathbf{U} + O\left(\frac{a}{R}\right)^3. \tag{5.2}$$

Using this relation and in addition (4.1), (4.2) and (5.1), we obtain, from (3.14) and (3.15) and using the appropriate relations for the acceleration in spherical polar coordinates, that to leading order in (a/R)

$$\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2 = \frac{36a^3U^2}{R^4}(3\cos^2\theta - 1), \tag{5.3}$$

$$R\frac{d^2\theta}{dt^2} + 2\frac{dR}{dt}\frac{d\theta}{dt} = \frac{72a^3U^2}{R^4}\cos\theta\sin\theta. \tag{5.4}$$

It is instructive to derive these relations in a more direct way. We start with (3.4) and take only the dipole $\mathbf{M}_1$ into account. For sphere A the strength of this is $(\mathbf{v}_B - \mathbf{U})a^3/2$. To leading order in (a/R) the velocity $\mathbf{u}_R$ induced in the centre of sphere B by the dipole in the centre of A, is

$$\mathbf{u}_R = \left\{\nabla\frac{\mathbf{U}\cdot a^3\mathbf{r}_1}{r_1^3}\right\}_{\mathbf{r}_1 = \mathbf{R}}. \tag{5.5}$$

With this, we deduce from (3.4), assuming the force on sphere B to be zero,

$$\frac{d}{dt}\{\tau\mathbf{v}_B - 4\pi a^3(\mathbf{v}_B - \mathbf{U})/2\} + 4\pi(\mathbf{U}a^3\cdot\nabla)\left(\nabla\frac{\mathbf{U}\cdot a^3\mathbf{r}_1}{r_1^3}\right)_{\mathbf{r}_1 = \mathbf{R}} = 0. \tag{5.6}$$

A similar expression can be formulated for sphere A. Subtracting these expressions results in

$$\frac{d}{dt}\{\tfrac{1}{2}(\mathbf{v}_A - \mathbf{v}_B)\tau\} + 8\pi a^3(\mathbf{U}\cdot\nabla_R)\nabla_R\left(\frac{\mathbf{U}\cdot a^3\mathbf{R}}{R^3}\right) = 0. \tag{5.7}$$

The subscript R in ∇_R denotes differentiation with respect to R. Making use of

$$\mathbf{v}_B - \mathbf{v}_A = \frac{d\mathbf{R}}{dt}, \tag{5.8}$$

and carrying out of the differentiation gives again the equations (5.3) and (5.4). It is interesting and important to note that these can be formulated in terms of a Lagrangian L given by

$$L = \frac{1}{2}\left\{\left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d\theta}{dt}\right)^2\right\} - \Omega, \tag{5.9}$$

with

$$\Omega = \frac{12a^3U^2}{R^3}(3\cos^2\theta - 1). \tag{5.10}$$

It is easily verified that (5.3) and (5.4) are the Euler equations of the Lagrangian variational principle applied to L. The quantity Ω plays the rôle of a potential energy. As explained in Section 1, the relative acceleration between spheres is completely due to the 'velocity squared' term in Bernoulli's equation. Obviously the associated force can in the present approximation be derived from a potential. There is therefore, a constant of the motion, Ω_0, say, given by

$$\Omega + \frac{1}{2}\left\{\left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d\theta}{dt}\right)^2\right\} = \Omega_0. \tag{5.11}$$

The sum of potential and kinetic energy remains constant for the relative motion because the mean velocity does not change in the present approximation. Writing

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \tag{5.12}$$

we may summarize (5.3) and (5.4) as

$$\frac{d}{dt}\mathbf{V} = \nabla\Omega. \tag{5.13}$$

We can derive from these equations a number of properties of the trajectories. Let us denote the initial values of θ and R with θ_0 and R_0, for some trajectory. Let us further denote with θ_c the value of θ such that

$$3\cos^2\theta_c - 1 = 0, \tag{5.14}$$

which means

$$\theta_c \sim 54°. \tag{5.15}$$

From (5.10) and (5.11) it follows that

$$\frac{3\cos^2\theta_0 - 1}{R_0^3} - \frac{3\cos^2\theta - 1}{R^3} \geqslant 0. \tag{5.16}$$

The initial value may be such that

$$3\cos^2\theta_0 - 1 < 0, \tag{5.17}$$

$$3\cos^2\theta_0 - 1 > 0, \tag{5.18}$$

$$3\cos^2\theta_0 - 1 = 0. \tag{5.19}$$

It suffices to restrict to values of θ between 0 and π because symmetry permits to extend the results to other values. We consider first initial values of θ obeying (5.17). This means, with a view to (5.14), that

$$\theta_c < \theta_0 < \pi - \theta_c, \tag{5.20}$$

a region denoted with I in Figure 3. For these values of θ_0 and at the same time with $R_0/R < 1$, it follows from (5.16) that

$$\theta_0 < \theta < \pi - \theta_0. \tag{5.21}$$

For values of θ_0 in the region (5.20) but with $R_0/R > 1$, θ may be everywhere in $\theta_c < \theta < \pi - \theta_c$. To decide on R_0/R we derive from (5.3), (5.4) and (5.11) that

$$\frac{d}{dt}\left(R\frac{dR}{dt}\right) = \frac{12a^3U^2}{R_0^3}\left\{(3\cos^2\theta_0 - 1) + \tfrac{1}{2}\left(\frac{R_0}{R}\right)^3(3\cos^2\theta - 1)\right\}. \tag{5.22}$$

This shows that for θ_0 obeying (5.17) and θ obeying (5.21), the right-hand side of (5.22) is negative. Hence, R decreases continuously.

We conclude therefore:

$$\theta_c < \theta_0 < \pi - \theta_c \Rightarrow \theta_c < \theta < \pi - \theta_c, \quad \text{and} \quad R_0/R > 1. \tag{5.23}$$

This is found in the numerical results also. The angle θ_c appears to be about 61° in the exact calculation. Next we consider values of θ_0 according to (5.18). To see how the relative motion develops after the initial moment, we first look at the angle θ. Since at $t = 0$ both dR/dt and $d\theta/dt$ are zero, it follows from (5.4) that θ starts to move towards $\theta = \pi/2$. Hence, from (5.22), R grows initially. If R grows so rapidly that $d\theta/dt$ becomes virtually zero before $\theta = \theta_c$ is reached (or $\pi - \theta_c$ if $\theta_0 > \pi - c$), then the asymptotic state is, as follows from (5.10) and (5.11),

$$\frac{dR}{dt} \sim (2\Omega_0)^{1/2}, \qquad \frac{d\theta}{dt} = 0. \tag{5.24}$$

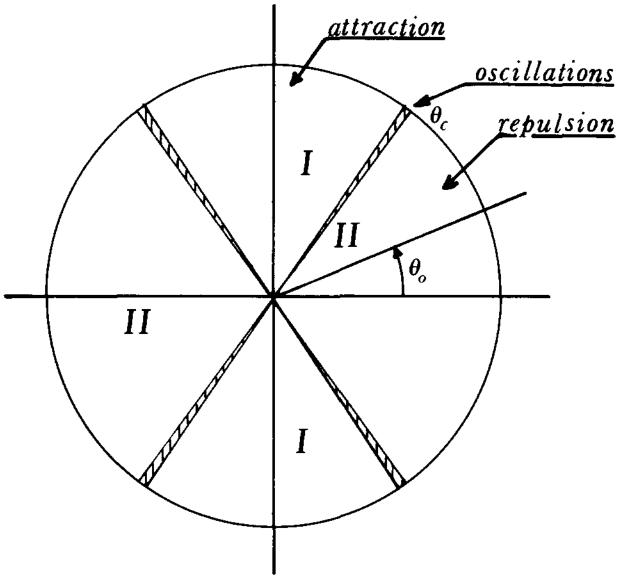

Figure 3. Results from approximate theory of Section 5 for relative motion between two spheres.

This is in complete agreement with the numerical results. Accordingly we may call most of the region where (5.18) applies the region of repulsion. This region is indicated with II in Figure 3. In contrast we might call region I the region of attraction. The asymptotic state (5.24) is reached by trajectories whose initial values obey (5.18), but not by all of these, however. When $(3\cos^2\theta_0 - 1)$ is very small but positive, or zero, the initial change of R is small. The angle θ, always starting to move towards $\theta = \pi/2$, may then increase beyond $\theta = \theta_c$ (or decrease below $\theta = \pi - \theta_c$). In that case R may both decrease or increase. In either case the change in R remains initially small. Trajectories starting in this narrow region adjacent to $\theta = \theta_c$ apparently agree with those resulting from the numerical computations in which the trajectory performs a number of oscillations before eventually ending up in region I or in region II. In this connection it is useful to point out that for $\theta_0 \approx \theta_c$, it is easy to derive from the equations of motion that for small times t

$$\theta = \theta_0 + O(t^2), \qquad R = R_0 + O(t^4).$$

This shows that for small times the rate of change of R is much smaller than the rate of change of θ.

6. Effect of relative motion on particle distribution

When one wishes to determine averages in a bubbly flow and one uses ensemble averaging, then the pair probability distribution is needed when interactions between a test bubble and one neighbour are considered. This distribution, giving the probability of finding a bubble at

$\mathbf{x} + \mathbf{R}$, given that there is one with centre in $\mathbf{x}$, is not a given quantity (which is the case with an inhomogeneous solid material for example) but follows from some initial value and the development of the flow. Batchelor and Green [11] have, for example, considered pair distribution functions in suspensions governed by viscous effects. Let us suppose that for $t < 0$, that is before the motion discussed in the preceding sections started, the bubbles are completely random distributed and have number density n. Then at $t = 0$, the pair probability function $P(\mathbf{R}, t)$ is given by

$$\begin{aligned} P(\mathbf{R}, 0) &= 0, \qquad \text{when} \qquad R < 2a, \\ P(\mathbf{R}, 0) &= n, \qquad \text{when} \qquad R > 2a. \end{aligned} \tag{6.1}$$

After $t = 0$ the bubbles start to move relative to each other as we have seen. This relative motion changes $P(\mathbf{R}, t)$, and we may ask for the eventual steady distribution, if existing. The evolution of $P(\mathbf{R}, t)$ is governed by (see for example Batchelor and Green [11])

$$\frac{\partial P}{\partial t} + \nabla\cdot(P\mathbf{V}) = 0, \tag{6.2}$$

where $\mathbf{V}$ is defined by (5.12). The above expression states that in $\mathbf{R}$, t-space the total number of pairs is conserved. When $\mathbf{V}$ is known, as solution in our case of (5.13), then P may be found from solving (6.2) with the initial condition (6.1). In the special case of the approximation of the previous section, the solution for P can be found without knowing $\mathbf{V}$. For this we observe first that, as follows from (5.7) and (5.10), Ω is a harmonic function satisfying Laplace's equation. Taking the divergence of both sides of equation (5.13) gives

$$\nabla\cdot\frac{d\mathbf{V}}{dt} = \nabla\cdot\frac{\partial\mathbf{V}}{\partial t} + \nabla\cdot\{(\mathbf{V}\cdot\nabla)\mathbf{V}\} = \nabla^2\Omega = 0. \tag{6.3}$$

Since $\mathbf{V}$ is irrotational we have

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla(\tfrac{1}{2}V^2). \tag{6.4}$$

The kinetic energy $\frac{1}{2}V^2$ satisfies Laplace's equation as well, which follows from $\nabla^2\Omega = 0$, (5.11) and (5.12). Hence from (6.3)

$$\frac{\partial}{\partial t}\,\nabla\cdot\mathbf{V} = 0. \tag{6.5}$$

Initially $\mathbf{V} = \mathbf{0}$ (remember that at $t = 0$ there is no relative velocity) and, therefore, $\nabla\cdot\mathbf{V}$ remains zero at $t > 0$, therefore, (6.2) reduces to

$$\frac{\partial P}{\partial t} + \mathbf{V}\cdot\nabla P = 0. \tag{6.6}$$

This means that the probability distribution remains constant along trajectories in **R**, t-space. Since initially we have supposed a completely random distribution, the latter continues to be so after $t = 0$. This is an important result because it relieves one from the task of calculating for each type of motion the associated distribution function. In the theory of suspensions governed by low-Reynolds-number effects, such calculations are requisite (see e.g., Batchelor and Green [11]). The question naturally arises whether the result (6.6) holds also for the specific problem of this paper. Inspection shows that this is not the case. The reason for this is that the relative motion cannot be derived from a Lagrangian in this specific problem, because energy is shared between the mean motion and the relative motion. However, we have shown that this effect is of order $(a/R)^9$. Hence to that order of accuracy it may be said that an initially random distribution remains so.

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